A Classical Approach to the Black-and-Scholes Formula and its Critiques, Discretization of the model

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1 Introduction

The public interest in the trading of stocks and other securities has continuously grown during the last decades. Opinions about future economic developments, events that influence stockprices and even the daily stock quotes and currency exchange rates receive higher and higher attention. The whole field as well as its underlying dynamics exercise a strong fascination. It seems therefore obvious to apply the tools of physics and mathematics to try to describe such complex systems.

One of the finest examples of such an description was presented in 1973 by Black and Scholes [1] and Merton [2] who provided a first reliable solution for the option-pricing problem. This model had a major impact on the trading of options. Before the introduction of the model there was no real market on options, also due to the fact that there was no suitable way of pricing them in a fair way. The Black-and-Scholes model changed the picture and presently options play an important role on the financial market. In 1997 Scholes and Merton received the Nobel prize in Economics for their works.

A classical approach to the Black-and-Scholes formula is shown in Chapter 2 of this paper. I also try to give some basic hints to the understanding of the underlying statistical concepts. The Black-and-Scholes formula is derived under a couple of assumptions that are also reason to question the correctness of the model. These critiques will be reflected in Chapter 3. Chapter 4 introduces a discretization of the model which is a major step to numerical simulation and it gives rise to an easier understanding of the Black-and-Scholes model.

2 The Black-and-Scholes Theory of Option Pricing

The major question of this chapter is how to determine the prices of derivative securities in particular options. Related to this we might have to measure risk involved in the investment into a derivative. The important achievement of Black and Scholes and Merton was to show that this is not necessarily true. Based on some assumptions on the price fluctuation of the underlying asset they could show that there is a dynamic hedging strategy for options by which risk can be eliminated completely.

2.1 Derivatives

Derivatives are financial instruments whose value depends on other, more underlying financial products which can be stocks, bonds, currencies etc. as well as commodities. A simple derivative is a so called forward contract (forward) which is a contract between two parties on the delivery of a certain asset at a fixed time in the future at a certain price. Forward contracts are usually not traded at exchanges. Standardized forward contracts which are offered at special exchanges are called future contracts (futures).

A more complex kind of derivatives are options. Unlike futures or forwards which have an obligation for both parties, options give the holder the right to buy or sell a certain asset in the future at a certain price. The writer still has the obligation to deliver or buy the underlying asset if the holder exercises the option.

Naturally there exists two types of options: call options, which give the holder the right to buy, and put options, which give the holder the right to sell an underlying asset at a certain price in the future. Options are distinguished as being of European type if the right to buy or sell can only be exercised at the date of maturity specified in the option, or of American type if they can be exercised at any time between the writing and their date of maturity. There exist other kinds of options besides these on the market.

2.2 Option Pricing in an Idealized Market

2.2.1 Assumptions

There are several assumptions on which the following derivation of the Black-and-Scholes Formula is based:

1. existence of a complete and efficient market
2. no transaction costs
3. all market participants can lend and borrow money at the same risk-free constant interest rate r
4. security trading is continuous
5. for simplification we assume that there will be no payoffs such as dividends from the underlying asset

2.2.2 Modeling Fluctuations of Financial Assets

The basic approach to the modeling of financial time series goes back to the work of Louis Bachelier and assumes a stochastic process. In a Markov stochastic process a next realization only depends on the present value of the random variable. The system has no longer-time memory, so non of the previous information of the series has any influence on the future outcome. In finance this process is reflected in the Efficient Market Hypothesis which states that all market participants quickly and comprehensively obtain all information relevant to trading. This means if there would be longer correlations all participants would have access to these data and would exploit them, which conversely counteract these correlations. A particular Markov process with a continuous variable and continuous time is the so called Wiener process which is usually used to model the behavior of stock prices. A discretization of a Wiener process should fulfill two basic properties, whereas the stochastic variable is called W:

1. Consecutive ΔW are statistically independent (ensures the Markov property).
2. ΔW is given for a small but finite time interval Δt by

   $$\Delta W = \varepsilon\sqrt{\Delta t} \tag{2.1}$$

   and for an infinitesimal interval dt by

   $$dW = \varepsilon\sqrt{dt} \tag{2.2}$$

   where ε is a random drawing from a normal distribution with a mean of zero and a standard deviation of 1.

We see that for a Wiener process the expectation value of the stochastic variable vanishes. Its variance is linear in Δt and the standard deviation behaves (for finite time intervals) as

$$\sqrt{\mathrm{var}(\Delta W)} = \sqrt{\Delta t} \tag{2.3}$$

The Wiener process may be generalized by superposing a drift dt onto the stochastic process dW

$$dx = a\,dt + b\,dW \tag{2.4}$$

which now has the following properties:

$$\langle x(T) - x(0)\rangle = aT \quad \text{and} \tag{2.5}$$

$$\mathrm{var}[x(T) - x(0)] = b^2 T \tag{2.6}$$

Applying equation (2.4) to the modeling of stock prices it seems obvious that it does not capture all the main features. A constant drift rate suggests that an expensive stock will on average make the same profit than a cheap one. This does not hold in reality. A better desciption would hold that the return of the investment is independent of the price of the asset. This can be written as

$$dS = \mu S dt \tag{2.7}$$

where μ is the return rate and S the price of the asset (stock). This has consequences for the risk of an investment, measured by the variance or, in financial contexts, the volatility of asset prices. A reasonable assumption is that the variance of the returns is independent of S. This means that in a time interval Δt

$$\mathrm{var}\left(\frac{\Delta S}{S}\right) = \sigma^2 \Delta t \qquad \text{or} \tag{2.8}$$

$$\mathrm{var}(S) = \sigma^2 \Delta t S^2 \tag{2.9}$$

Equation (2.4) can now be rewritten as

$$dS = \mu S dt + \sigma S dW \quad \text{or} \quad \frac{dS}{S} = \mu dt + \sigma \varepsilon \sqrt{dt} \tag{2.10}$$

which represents a so called Itô process. It is often refered to as geometric Brownian motion and is the most widely used model of stock price behavior. The model must still be considered a hypothesis which has to be checked critically. It will be used as the model describing the behavior of the underlying in the derivation of the Black-and-Scholes formula and shall be considered a further assumption.

The Itô Lemma

Describing the price of an asset following equation (2.10) one has to know the properties of functions of stochastic variables. An important result in this area, that we need for the further development, is a lemma due to Itô. Let $x(t)$ follow an Itô process

$$dx = a(x,t)dt + b(x,t)dW = a(x,t)dt + b(x,t)\varepsilon\sqrt{dt} \tag{2.11}$$

Then a function $G(x,t)$ of the stochastic variable x and the time t also follows an Itô process given by

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial x^2}\right)dt + b\frac{\partial G}{\partial x}dW \tag{2.12}$$

The drift of this process is given by the first term on the right-hand side in parentheses, and the standard deviation rate is given by the prefactor of dW in the second term.
An approach to the Itô Lemma was presented in class.

2.2.3 Classical Option Pricing

Investment in options is usually considered risky. It also seems unfair that the writer of an option engages in a liability when entering the contract, while the holder has a freedom of action depending on market movements. The questions are, what is the risk premium for the writer of the option, what is the price for the freedom of the holder? What is the value of the asymmetry of the contract?
These questions were answered by Black, Merton and Scholes ([1] and [2]) under the assumptions specified above: *There is no risk premium required for the writer.* The price of the option is determined completely by the volatility of the stock and the conditions of the contract.

The following discussion will be limited to European options. Assuming that the price of the underlying stock S of an option follows (compare (2.10))

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2.13}$$

where μ is the expected return per unit time and σ the volatility (intensity of fluctuation) of the stock price. The expected return rate μ already includes a risk premium and is therefore greater then the risk-free interest rate r. The value of an option Ω depends on the price of the underlying asset as well as on time

$$\Omega = \Omega(S(t), t) \tag{2.14}$$

Using Itô's Lemma we can derive

$$d\Omega = \left(\frac{\partial \Omega}{\partial S} \mu S + \frac{\partial \Omega}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \Omega}{\partial S^2} \right) dt + \sigma S \frac{\partial \Omega}{\partial S} dW \tag{2.15}$$

The main idea underlying the works of Black, Merton and Scholes is that it is possible to construct a riskless portfolio composed of the option and the underlying security. Being riskless, this portfolio can only earn the risk-free interest rate r. The formation of such a riskless portfolio is possible because both the stock and the option price depend on the same source of uncertainty, namely the same stochastic process. This stochastic process can be eliminated by a suitable linear combination of both assets.
The dependence of the option price Ω on that of the underlying is given by $\Delta = \partial\Omega/\partial S$. Taking the position of a writer in a European call our portfolio will be composed of (i) a short position in one call option and (ii) a long position in $\Delta(t) = \partial\Omega/\partial S$ units of the underlying which has to be adjusted continuously with the stock price. The value of our portfolio Π is

given by

$$\Pi(t) = -\Omega(t) + \frac{\partial \Omega}{\partial S} S(t) \tag{2.16}$$

and it follows the stochastic process (using (2.13) and (2.15))

$$d\Pi = -d\Omega + \frac{\partial \Omega}{\partial S} dS = \left(-\frac{\partial \Omega}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Omega}{\partial S^2} \right) dt \tag{2.17}$$

Being riskless the portfolio must earn the risk-free interest rate r

$$d\Pi = r\Pi dt = r\left(-\Omega + \frac{\partial \Omega}{\partial S} S \right) dt \tag{2.18}$$

Equating (2.17) and (2.18), we obtain

$$r\Omega = \frac{\partial \Omega}{\partial t} + rS\frac{\partial \Omega}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Omega}{\partial S^2} \tag{2.19}$$

which is the Black-and-Scholes partial differential equation. It does not involve any variables that are affected by the risk preferences of investors, which is the most important tool in the analysis of derivative securities.

So far no assumtions have been made about the specific kind of options. Formula (2.19) is valid for European calls as well as puts. In order to obtain solutions for the Black-and-Scholes equation we have to specify boundary conditions. At maturity T the prices of the call C and the put P are

$$\begin{aligned} \text{Call:} \qquad C &= \max(S(T) - X, 0) \\ \text{Put:} \qquad P &= \max(X - S(T), 0) \end{aligned} \tag{2.20}$$

where X is the strike price of the option. Following the approach by Black and Scholes we substitute

$$\begin{aligned} \Omega(S,t) &= e^{-r(T-t)} y(u,v)\,, \\ u &= \frac{2\rho}{\sigma^2}\left(\ln\frac{S}{X} + \rho(T-t) \right), \\ v &= \frac{2\rho^2}{\sigma^2}(T-t)\,, \qquad \rho = r - \frac{\sigma^2}{2} \end{aligned} \tag{2.21}$$

This reduces equation (2.19) to a $1D$ diffusion equation:

$$\frac{\partial y(u,v)}{\partial v} = \frac{\partial^2 y(u,v)}{\partial u^2} \tag{2.22}$$

Diffusion equations are solved by Fourier transform reducing (2.22) to an ordinary differential equation in v [3], [4]. The final solution for the Black-and-Scholes equation for a European call option is given by

$$C(S,t) = S\,N(d_1) - Xe^{-r(T-t)}N(d_2)\,. \tag{2.23}$$

$N(d)$ is the cumulative normal distribution

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} dx\, e^{-x^2/2}\,, \tag{2.24}$$

and the two arguments are given by

$$d_1 = \frac{\ln\frac{S}{X} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{2.25}$$

$$d_2 = \frac{\ln\frac{S}{X} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{2.26}$$

Using the put-call-parity [4]

$$C(t) + Xe^{-r(T-t)} = P(t) + S(t) \tag{2.27}$$

one can derive the price of a put option to be

$$C(S,t) = -S[1 - N(d_1)] + Xe^{-r(T-t)}[1 - N(d_2)]\,. \tag{2.28}$$

The Black-and-Scholes formulas (2.23) and (2.28) tell the writer what price she should charge for an option at time t. The price depends on the parameters X and T of the contract and on the market characteristics r and σ.

Additionally the equations provide the necessary information to eliminate risk. The writer's portfolio only stays risk-less if he continuously adjusts the amount of underlying $\Delta(t)$. This strategy is called Δ-hedging. For a call option this gives [4]

$$\Delta(t) = \frac{\partial C}{\partial S} = N(d_1) \qquad (0 \le N(d_1) \le 1) \tag{2.29}$$

The different terms in (2.23) have an immediate interpretation [3] if a term $e^{-r(T-t)}$ is factored out:

1. $N(d_2)$ is the probability of the exercise of the option in a risk-neutral world, i.e. where the actual drift of a financial time series can be replaced by the risk-free rate r.

2. $X\,N(d_2)$ is the strike price times the probability that it will be paid, i.e. the expected amount of money to be paid under the option contract.

3. $S\,N(d_1)\,e^{r(T-t)}$ is the expectation value of $S(T)\,\Theta(S(T)-X)$ in a risk-neutral world.

4. The difference of this term with $X\,N(d_2)$ is then the profit expected from the option. The prefactor $e^{-r(T-t)}$ does nothing else than discounting this profit down to the present day value. This is precisely the option price.

The one parameter in the Black-and-Scholes equation that can not be observed directly is the volatility σ of the underlying asset. Estimating the volatility is not a straightforward procedure [5]. There are several approaches to get information about volatility. The use of historical data is one way although volatility measured over long terms might be quite different from the volatility observed during the lifetime of the option. A more commonly used way is to measure implied volatility. This is done by using the Black-and-Scholes formula backwards, taking present option prices and calculating the volatility other option traders expect for the future.

The Black-and-Scholes model is an elegant framework to understand and model ideal financial markets. Due to the assumptions made to derive the model it provides only an approximate description of the real world. These problems will be discussed in the next chapter.

3 Critiques of the Black-and-Scholes Model

Option trading involves risk! (disclaimer found on most Chicago Board Options Exchange documents [6])

Dealing with the Black-and-Scholes formula one has to keep in mind that the model only holds under the assumptions stated in the previous chapter. In fact many of them are violated in the real world and demand corrections or generalizations.

The Black-and-Scholes approach is based on the existence of a perfect hedging strategy to keep a certain portfolio riskless in time. Following this theory the Δ-hedge within that portfolio has to be adjusted continuously. But even for large financial institutions there exist transaction costs that do not allow a continuous hedging. Also there is naturally a discretness in transactions and moreover it is widely common to trade financial assets in round lots of 100 which violates the implied assumption of indivisibility [5]. This argumentation shows that there does not exist a completely riskless hedging strategy in the real market and that there will be a finite risk premium on options. On the other hand this is the reason *why option markets exist at all!* The market is based on that remaining uncertainty which makes trading attractive. A french group with J. Bouchaud [6] based an approach to the theory of option pricing on a global wealth balance that in the first place allows the existence of risk. Their theory only tries to minimize risk and is therefore far more general. The Black-and-Scholes formula then arises in the limit of zero risk. Their interpretation corresponds to developments already visible in economics - the establishment of risk-management that does not deny the fact of existing risk.

Besides the stated non-existence of a riskless portfolio discrete transaction steps also result in some correlations which may persist over at least a couple of minutes on the trading floor. In that respect geometric Brownian motion only occurs in the mathematical limit for continuous trading [6] assuming a Markov process. Further objections against the used statistics were raised by Mandelbrot [7]. He showed that extreme events which occur in in stock price statistics are underestimated in the description by geometric Brownian motion. In the same sense the model also does not take discontinuous behavior of asset prices into account caused by information from politics, natural disasters etc. which have a strong impact on the economical data.

The Black-and-Scholes model also assumes the interest rate r and the volatility of stocks σ being constant. There exist formalisms to take variations of these parameters in account (see [3]). Still the problem of measuring and estimating volatility is present in real world markets as explained in the previous chapter. But also these uncertainties, including the different measurements of each trader, contribute to the existence of the market.

Besides all the simplifications used in the derivation of the Black-and-Scholes formula the model is still the most widely used instrument in option trading. As long as one is aware of the shortcomings of the model it provides substantial information on option markets.

4 Discretization of the Model

A discretization of the Black-and-Scholes model leads to the introduction of discrete timesteps which themselves demand discrete statistical events. A common approach is the evaluation of option values in so called binary trees first introduced in 1979 ([8] and [9]). For a non-path-dependent option, like a European one which only depends on the their value at maturity, the binary model converges to the result of the Black-and-Scholes model [8]. On the other hand the Black-and-Scholes formula is not suitable for the evaluation of path-dependent options like American options. This can be done using appropriate binary models.

The price of a stock at time t is $S(t)$. After a discrete timestep Δt the price is either $u \cdot S(t)$ or $d \cdot S(t)$ depending if the price goes up (u) or down (d). The values of u and d represent the relative change in the stockprice after the interval Δt. With a (pseudo)-probability $\overline{p}$ the value of an option increases in the timestep Δt, with (pseudo)-probability $\overline{q}$ it decreases. The development of the value of a call option after a timestep Δt can then be estimated by

$$\overline{p}(u \cdot S(t) - X) + \overline{q}(d \cdot S(t) - X) = S(t) - e^{-\mu\Delta t}X \tag{4.1}$$

where X is the strike price of the option and μ expected rate of return which under the assumption of the Black-and-Scholes model is nearly the riskfree interest rate r. Using

$$z = e^{\mu\Delta t} \tag{4.2}$$

this can be rewritten as

$$\overline{p}(u \cdot S(t)) + \overline{q}(d \cdot S(t)) - (\overline{p} + \overline{q})X = S(t) - (1/z)X\,. \tag{4.3}$$

Comparison of coefficients yields

$$\begin{aligned} \overline{p} \cdot u + \overline{q} \cdot d &= 1 \qquad \text{and} && (4.4)\\ \overline{p} \cdot z + \overline{q} \cdot z &= 1\,. && (4.5) \end{aligned}$$

It follows that $\overline{p}+\overline{q} \neq 1$ which legitimates the expression pseudo-probabilities. Futhermore $\overline{p}$ and $\overline{q}$ can now be expressed in terms of u, d and z.

$$\begin{aligned} \overline{p} &= \frac{z-d}{z(u-d)} && (4.6)\\ \overline{q} &= \frac{u-z}{z(u-d)} = \frac{1}{z} - \overline{p} && (4.7) \end{aligned}$$

This result is important for the construction of discrete option pricing models. The value of an option Ω at time t is now given by

$$\Omega(S(t),t) = \max[\overline{p} \cdot \Omega(u \cdot S(t), t+\Delta t) + \overline{q} \cdot \Omega(d \cdot S(t), t+\Delta t), \Sigma(S(t),t)]\,. \tag{4.8}$$

In this equation $\Sigma(S(t), t)$ represents the profit when exercising the option at time t (if this possibility exists). The first term in (4.8) refers to the expected development of the option value Ω in the future. Interpreting it for an American option it is just an evaluation of the option after each timestep Δt. If the profit Σ for exercising the option right now is greater then its expected future value then the holder exercises it. At maturity T the option value Ω equals the profit Σ

$$\Omega(S(T), T) = \Sigma(S(T), T) \tag{4.9}$$

The distribution of the stock prices given by the binomial model converges onto a log-normal distribution [8]. This is the same behavior if the underlying dynamics of stock prices are described by geometric Brownian motion [3] which was motivated in chapter 2. This allows to establish a correlation between u and d and the volatility σ of the stock on which the option contract is based. A suitable choice of the binomial pricing parameters as a function of the discrete time interval Δt and the volatility σ are ([10], [11])

$$u = e^{\sigma\sqrt{\Delta t}}, \tag{4.10}$$

$$d = e^{-\sigma\sqrt{\Delta t}}. \tag{4.11}$$

These are the main aspects of the discrete model. The idea in numerical simulations is to calculate all the different possibilities of the stock price development, given by the pseudo-probabilities $\overline{p}$ and $\overline{q}$, until maturity of the option. Then an evaluation using (4.9) can be performed. Because at each discrete timestep also formula (4.8) can be used, an evaluation for other options besides European options is possible as well.

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